

EMBEDDINGS AND IMMERSIONS OF TROPICAL CURVES

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ABSTRACT. We construct immersions of trivalent abstract tropical curves in the Euclidean plane and embeddings of all abstract tropical curves in higher dimensional Euclidean space. Since not all curves have an embedding in the plane, we define the tropical crossing number of an abstract tropical curve to be the minimum number of self-intersections, counted with multiplicity, over all its immersions in the plane. We show that the tropical crossing number is at most quadratic in the number of edges and this bound is sharp. For curves of genus up to two, we systematically compute the crossing number. Finally, we use our immersed tropical curves to construct totally faithful nodal algebraic curves via lifting results of Mikhalkin and Shustin.

1. INTRODUCTION

Tropical geometry has been studied in two different flavors. One is the abstract version that views a tropical variety as a skeleton of a Berkovich analytification of an algebraic variety defined over a non-archimedean field. The second is the embedded version that studies tropicalization of a variety embedded in the algebraic torus or in projective space. In this paper, we examine the combinatorial relationship between these two different views. First, we show that an abstract tropical curve can always be represented as an embedded one:

Theorem 1.1. *Let Γ be an abstract tropical curve and suppose that d is the largest degree of a vertex of Γ . Then Γ has a smooth embedding in \mathbb{R}^n when n is at least $\max\{3, d - 1\}$ and Γ has an immersion in \mathbb{R}^2 if d is at most 3.*

The conditions in Theorem 1.1 are as sharp as possible because the local model for a degree d vertex only embeds in \mathbb{R}^{d-1} , so it is not possible to have an embedding or immersion in \mathbb{R}^n when $n < d - 1$. Moreover, while some abstract tropical curves can be embedded in \mathbb{R}^2 , some cannot, for example, because there might not even be an embedding of the underlying graph. However, by Theorem 1.1, we always have an immersion of an abstract tropical curve in \mathbb{R}^2 and we define the **tropical crossing number** to be the

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minimum number of crossings, counted with tropical multiplicities, over all possible planar immersions. The tropical crossing number is always bounded below by the graph-theoretic crossing number of the underlying graph, which is the minimum number of crossings among all planar immersions of the graph. Our proof of the existence of immersions also bounds the tropical crossing number:

Theorem 1.2. *If Γ is a trivalent abstract tropical curve with e edges, then the tropical crossing number of Γ is at most $O(e^2)$.*

The quadratic bound in Theorem 1.2 is optimal up to a constant factor, because the graph-theoretic crossing number of a random cubic graph grows quadratically in the number of edges [RS09, p. 143–144]. However, we show that the gap between the two crossing numbers can itself grow quadratically, for an explicit family of tropical curves:

Theorem 1.3. *There exists a family of trivalent abstract tropical curves with e edges, whose underlying graph is planar, but whose crossing number is $\Theta(e^2)$.*

The family in Theorem 1.3 is the chain of loops with bridges and generic edge lengths as in [JP14], where the bridges are necessary so that the graph is trivalent and therefore has a finite crossing number. We use the result from [CDPR12] that these abstract tropical curves are Brill-Noether general, or, more specifically, have the divisorial gonality of a Brill-Noether general curve. Any such family of abstract tropical curves would also have quadratic tropical crossing number.

Having looked at the asymptotic behavior of tropical crossing numbers for large graphs, we then turn to the small graphs, specifically of genus at most 2. Perhaps surprisingly, in Proposition 4.7, we give a case of an abstract tropical curve which has a planar embedding for generic values of the metric parameters, but not for specializations. As a consequence, the crossing number is neither lower nor upper semi-continuous in the metric parameters (see Remark 4.8).

A complementary analysis of curves with tropical crossing number 0, going up to genus 5 has been undertaken in [BJMS15]. They describe the closure of the locus of curves with crossing number 0 inside the moduli space of all curves, in terms of equalities and inequalities on the metric parameters of any given graph. Their techniques are computational and demonstrate that there are effective and practical algorithms for classifying abstract tropical curves with a given crossing number.

Using the realizability results of Mikhalkin and Shustin, we have the following application to algebraic curves and their analytifications.

Theorem 1.4. *Let K denote the field of convergent Puiseux series with complex coefficients. If Γ is a trivalent abstract tropical curve with rational edge lengths, then there exists a K -curve C , such that the minimal skeleton of*

the Berkovich analytification C^{an} is isometric to Γ , and there is an immersion from an open subset $C' \subset C$ to \mathbb{G}_m^2 with totally faithful tropicalization.

The field of convergent Puiseux series in Theorem 1.4 refers to the subfield of the field of formal Puiseux series $\mathbb{C}\{\{t\}\}$ consisting of those series which converge for sufficiently small values of t . Thus, the K -curve from Theorem 1.4 can be explicitly specialized to a curve over \mathbb{C} by choosing a sufficiently small value of t .

The totally faithful tropicalization appearing in Theorem 1.4 is a strengthening of the faithful tropicalizations studied in [BPR11] and [GRW14], but for nodal curves rather than smoothly embedded curves. Baker, Payne, and Rabinoff proved that for any algebraic curve C and a skeleton of its analytification C^{an} , there exists an embedding of an open dense subset $C \supset C' \rightarrow \mathbb{G}_m^n$, such that the projection of $(C')^{\text{an}}$ to its tropicalization is an isometry on its skeleton [BPR11, Thm. 1.1]. Totally faithful tropicalizations were defined in [CFPU14] as the strengthening where the tropicalization is required to be an isometry also on the skeleton of C' , which contains additional unbounded edges for every point in $C \setminus C'$. We work with the nodal version of a totally faithful tropicalization, where $C' \rightarrow \mathbb{G}_m^2$ is only an immersion, and the tropicalization correspondingly also has nodes. The analogue of Theorem 1.4 for embeddings in higher dimensions appears in [CFPU14], and combines our results on embeddings of tropical curves with their work on realizations of such curves.

The rest of the paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2 on the existence of embeddings and immersions of tropical curves. Section 3 proves Theorem 1.3 by relating the crossing number to the divisorial gonality. Section 4 contains a study of crossing numbers for graphs of genus at most 2, and Section 5 proves Theorem 1.4, applying our results to realizability questions for algebraic curves.

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2. EMBEDDINGS AND IMMERSIONS OF TROPICAL CURVES

In this section, we present the construction of the smooth planar immersion and embedding of tropical curves in \mathbb{R}^n . We start by recalling the definitions

of abstract and embedded tropical curves, as well as the relation between the two.

Definition 2.1. An **abstract tropical curve** is a finite connected graph possibly with loops or multiple edges together with either a positive real number or infinity attached to each edge, which will be known as the **length** of the edge. Any edge with infinite length must have a degree 1 vertex at one of its endpoints, which will be referred to as an **infinite vertex**.

A **subdivision** of an abstract tropical curve consists of replacing an edge with two consecutive edges whose lengths add up to the length of the original edge. If the original edge was infinite then the subdivided edge incident to the infinite vertex must also be infinite. Two tropical curves are **equivalent** if one can be transformed into the other by a series of subdivisions and reverse subdivisions. \square

Our definition of an abstract tropical curve is based on the one in [Mik05], but slightly more general because we do not require all 1-valent vertices to be infinite. Because of this, our definition is equivalent to that in [ABBR15, Sec. 2.1].

Remark 2.2. The underlying graph of a tropical curve has a natural realization as a topological space and the lengths along the edges additionally give a metric on this realization, away from the infinite vertices. This metric realization gives an alternative characterization of abstract tropical curves up to equivalence as inner metric spaces which have a finite cover by open sets isometric to star shapes. See, for example, [ABBR15, Sec. 2.1], for a definition from this perspective. \square

We will also consider a coarsening of the above equivalence for tropical modification. An **elementary tropical modification** is formed by adding an infinite edge at a finite vertex. A **tropical modification** is any sequence of elementary tropical modifications, subdivisions, and reverse subdivisions.

Example 2.3. In the center of Figure 1 is an abstract tropical curve, which is a tropical modification of the tropical curve on the right of that figure. This tropical modification is obtained by first subdividing each of the edges of the latter graph into 4 edges of equal length and then attaching infinite edges to 7 of the 9 newly created vertices. In Example 2.7, we will see that the leftmost diagram in Figure 1 gives an immersion of the modified graph. \square

We now turn to the embedded side and define smooth and nodal tropical curves in \mathbb{R}^n . For any integer d in the range $2 \leq d \leq n + 1$, the standard smooth model of valence d is the union of the d rays generated by the coordinate vectors e_1 through e_{d-1} and the vector $-e_1 - \cdots - e_{d-1}$. Up to changes of coordinates in $GL_n(\mathbb{Z})$ these are exactly the 1-dimensional matroidal fans from [AK06] and such fans form the building blocks of tropical manifolds [MZ14, Def. 1.14].

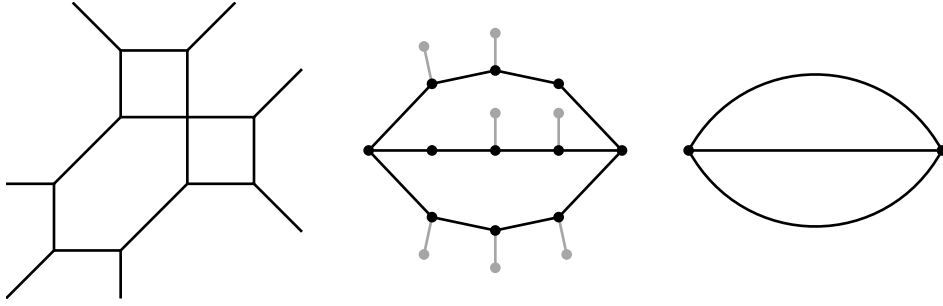


FIGURE 1. The immersed tropical curve on the left gives rise to the abstract tropical curve in the center. The vertices and edges shown in gray are infinite and the other edges all have finite lengths, which happen to be equal. This abstract tropical curve is a tropical modification of the curve on the right, in which all edge lengths are equal to 4 times the finite edge lengths of its tropical modification. We will refer to the underlying graph of this last curve as a theta graph.

Definition 2.4. A **smooth tropical curve** in \mathbb{R}^n will be a union of finitely many segments and rays such that a neighborhood of any point is equal to a neighborhood of a standard model, after a translation and a change of coordinates taken from $GL_n(\mathbb{Z})$. \square

For nodal curves, we have an additional local model. Recall that in algebraic geometry, a nodal curve singularity is one that is analytically isomorphic to a union of two distinct lines meeting at a point. For tropical curves, the local model for a node will analogously consist of two distinct (classical) lines with rational slopes, passing through the origin in \mathbb{R}^2 . We will only ever consider nodal curves in the plane.

Definition 2.5. A **nodal tropical curve** in \mathbb{R}^2 is a union of finitely many segments and rays which is locally equal to either to a standard smooth local model or a nodal local model, again up to translation and the action of $GL_2(\mathbb{Z})$. \square

The embedded and abstract tropical curves are related in that any embedded curve gives rise to an abstract one, as we now explain. By definition, each smooth tropical curve is a union of segments and rays, so if we add a vertex “at infinity” for each unbounded ray, we naturally get a finite graph. Thus, it only remains to assign lengths to the edges of this graph. In the standard local model, since we only allow changes of coordinates in $GL_n(\mathbb{Z})$, any segment is parallel to an integer vector. In other words, if p_1 and p_2 are the endpoints of a segment, then $p_1 - p_2 = \alpha v$, where α is a positive real number and v is a primitive integral vector i.e., $v \in \mathbb{Z}^n$ and the entries have no common divisor. Then α is uniquely determined and we use it as the

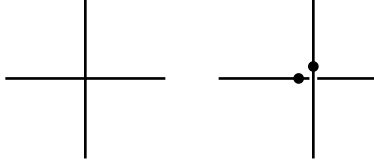


FIGURE 2. Procedure for resolving the nodal singularities in a plane curve. On the left is a neighborhood of a node in an embedded tropical curve and on the right, its resolution as an abstract tropical curve, which includes two vertices and the ends of four edges.

length of the edge. This is the same metric used in Mikhalkin's enumerative results [Mik05, Rmk. 2.4].

For nodal curves in \mathbb{R}^2 , we use the procedure as above for edges and smooth vertices. For the local model consisting of two lines passing through the origin, the abstract tropical curve has two vertices which map to the origin, one of which is an endpoint for the edges corresponding to one line and the other corresponding to the other line. This is illustrated in Figure 2.

Definition 2.6. Let Γ be an abstract tropical curve. An **embedding** in \mathbb{R}^n (resp. an **immersion** in \mathbb{R}^2) is a smooth (resp. nodal) tropical curve in \mathbb{R}^n (resp. \mathbb{R}^2) which is isometric to a tropical modification of Γ . \square

Example 2.7. The leftmost curve of Figure 1 is a nodal tropical curve with a single node. After resolving the node, we obtain the abstract tropical curve in the center, so the leftmost curve is an immersion of the center curve. Although the segments in the immersed curve have different lengths in the Euclidean metric, each finite edge of the realized graph has the same length.

Moreover, as we saw in Example 2.3, the center abstract tropical curve is a tropical modification of the curve on the right, and thus the leftmost curve is also an immersion of the rightmost abstract tropical curve. \square

We define the notion of multiplicities and then present a proof of Theorems 1.1 and 1.2. Recall from Definition 2.5 that a neighborhood of a nodal point of a tropical curve is the union of two lines. As in the construction of the edge lengths, we can assume that these lines are parallel to integral vectors $v = (v_1, v_2)$ and $u = (u_1, u_2)$ respectively, and that these integral vectors are primitive. Then the multiplicity of this node is

$$(1) \quad \left| \det \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix} \right|$$

This determinant is equivalent to the multiplicity of the stable intersection of the two lines, such as in [MS15, Def. 3.6.5]. We now prove our main theorems of this section.

Proof of Theorems 1.1 and 1.2. Fix $n \geq 2$ and let Γ be a graph whose vertices have degree at most $n + 1$. We will construct an embedding (if $n \geq 3$) or an immersion (if $n = 2$) of Γ , as follows.

If Γ has any loops or parallel edges, we first subdivide them to obtain an equivalent graph which does not have loops or parallel edges. We then label each vertex-edge incidence of Γ with an integer from 0 to n inclusive such that the labels are distinct at each vertex, and along each edge, the labels at the endpoints differ by ± 1 , modulo $n + 1$. The first part of this condition can always be achieved since the degree of any vertex is at most $n + 1$ and if the second part is violated at an edge, then we can resolve it by subdividing that edge sufficiently many times.

Fix a sufficiently small real number L such that nL is less than the length of any edge of Γ and we will construct most of the embedding inside the n -dimensional cube $[0, L]^n$. We begin by defining $\iota: \Gamma \rightarrow \mathbb{R}^n$ on the finite vertices of Γ by sending them to points inside this n -dimensional cube, and we will later assume that these points are generic.

To extend ι to a small neighborhood of each vertex, we let e_1, \dots, e_n denote the basis elements $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, \dots , $(0, 0, \dots, 1)$ of \mathbb{Z}^n and set $e_0 = (-1, \dots, -1)$. In a neighborhood of a vertex v , we send a small interval along each edge incident to v to an interval in the direction e_i if i is the label on the edge-vertex pair. In order to complete the local model, we add infinite edges at v until its degree is $n + 1$ and then send these infinite edges to rays in the direction e_i for the indices i which were not among the labels around v . We also send the infinite edges of Γ to rays in the already determined directions.

We now extend the map ι to the finite edges of Γ as follows. For each edge E between vertices u and w in Γ , we already defined ι on neighborhoods of each endpoint, thus giving the initial tangent directions. We adopt the convention that the index i for the vector e_i is taken modulo $n + 1$, so $e_{n+1} = e_0$, and so on. Thus, we can assume that the tangent directions at u and w are e_i and e_{i+1} respectively. Since e_i, \dots, e_{i+n-1} form a basis for \mathbb{R}^n , we can uniquely write:

$$\iota(w) - \iota(u) = \alpha_0 e_i + \alpha_1 e_{i+1} + \dots + \alpha_{n-1} e_{i+n-1}$$

We set $m = |\alpha_0| + \dots + |\alpha_{n-1}|$, and let ℓ be the length of the edge E . Recall that $\iota(w)$ and $\iota(u)$ are inside a box of side length L , so $m \leq nL < \ell$, by our choice of L .

We now let $\alpha'_0, \alpha''_0, \alpha'_1$, and α''_1 be the unique numbers such that for $i = 0, 1$, we have $\alpha'_i > 0$, $\alpha''_i < 0$, $\alpha'_i + \alpha''_i = \alpha_i$, and

$$|\alpha'_i| + |\alpha''_i| - |\alpha_i| = (\ell - m)/2.$$

Then, we can map the edge E to connect $\iota(u)$ and $\iota(w)$ in a piecewise linear fashion, with all segments parallel to one of e_i, \dots, e_{i+n-1} . In particular, we

define $\iota(E)$ to linearly interpolate between the following points:

$$\begin{aligned}
& \iota(u) \\
& \iota(u) + \alpha'_0 e_i \\
& \iota(u) + \alpha'_0 e_i + \alpha'_1 e_{i+1} \\
& \iota(u) + \alpha'_0 e_i + \alpha'_1 e_{i+1} + \alpha''_0 e_i \\
& \iota(u) + \alpha'_0 e_i + \alpha'_1 e_{i+1} + \alpha''_0 e_i + \alpha_{n-1} e_{i+n-1} \\
& \iota(u) + \alpha'_0 e_i + \alpha'_1 e_{i+1} + \alpha''_0 e_i + \alpha_{n-1} e_{i+n-1} + \alpha_{n-2} e_{i+n-2} \\
& \vdots \\
& \iota(u) + \alpha'_0 e_i + \alpha'_1 e_{i+1} + \alpha''_0 e_i + \alpha_{n-1} e_{i+n-1} + \alpha_{n-2} e_{i+n-2} + \cdots + \alpha_2 e_{i+2} \\
& \quad = \iota(w) - \alpha''_1 e_{i+1} \\
& \iota(w)
\end{aligned}$$

The constants α'_i and α''_i are chosen such that this path has total length ℓ , which is the desired edge length, and such that the tangent directions from $\iota(u)$ and $\iota(w)$ are e_i and e_{i+1} , respectively. In order to complete the local model at each vertex other than the endpoints $\iota(u)$ and $\iota(v)$, we need to add an unbounded ray in the appropriate direction, corresponding to a subdivision and modification of Γ , which we will call Γ' .

To show that ι defines an immersion for $n = 2$, we need to show that for generic choices of images for the vertices of Γ , the self-intersections of ι will only occur at edges. If w is a vertex of Γ , then a small perturbation of $\iota(w)$ will move $\iota(w)$ off of any other point of $\iota(\Gamma')$, even if that point is in an edge of Γ containing w . On the other hand, if v is a vertex of Γ' formed by the subdivision of Γ , then the key observation is that both coordinates of $\iota(v)$ will be affected by at least one of the endpoints of the original edge e on which v occurs. Moreover, even if $\iota(v)$ is contained in an edge of Γ sharing an endpoint with e , then the two edges start in different directions from their endpoints, so, in Γ' , the vertex v is not adjacent to the common endpoint, so we can perturb $\iota(v)$ using the other endpoint of e . Since Γ has no parallel edges, this means that for any intersection of $\iota(v)$ with another edge, a small perturbation of one of the vertices of Γ will push $\iota(v)$ off of that edge without affecting the edge. By keeping the perturbation sufficiently small, we will not introduce any new self-intersections, and thus we can obtain the desired immersion.

For $n \geq 3$, we will show that for generic choices for an embedding of the vertices, ι is injective. Here, assume that we have an intersection between two different embedded edges. By our initial subdivisions, these edges share at most one endpoint in common. Moreover, by our choice of the directions for the edge near the endpoints, even if the edges share an endpoint v , the intersection will not be in the first two segments next to v . Thus, at least two of the coordinates of this point of intersection will depend on the

opposite endpoint, so there is at least a 2-dimensional space of perturbations that will move the segment. On the other hand, the perturbations parallel to the other segment will still have a self-intersection, but this will be at most a 1-dimensional subspace. Thus, we can find some perturbation which eliminates the intersection.

Finally, we need to prove the quadratic bound on the number of crossings in the case of immersions in \mathbb{R}^2 . The subdivisions to avoid loops or parallel edges can be done by replacing each edge by 2 or 3 edges respectively. Then, our immersion further subdivides each edge into 4 segments and introduces 3 infinite rays. Thus, each edge of Γ results in a bounded number of segments in the immersion. Moreover, these segments are each parallel to one of the vectors e_0 , e_1 , and e_2 , so the intersection multiplicity of any two is at most one. Thus, the total number of crossings of ι is $O(e^2)$, as desired. \square

Remark 2.8. We note that the embeddings and immersions constructed in the proof of Theorem 1.1 have the following additional properties, which will be relevant for the applications to realizability of curves in Theorem 1.4 and in [CFPU14]. First, for every edge of Γ , the directions of the embeddings of the edges in the subdivision form a basis for \mathbb{Z}^n . Second, if all of the lengths of Γ are rational, then all the vertices of the tropical curve can also be chosen to be rational. Third, vertices contained in three or more bounded edges are only adjacent to vertices contained in at most two bounded edges, because every edge of the original graph is subdivided. \square

Remark 2.9. In algebraic geometry and other areas, immersions and embeddings may be constructed by starting with an embedding in a high-dimensional space and projecting (for example, Prop. IV.3.5 and Thm. IV.3.10 in [Har77]). However, the key to such arguments is showing that generic projections preserve embeddings or create immersions. However, in tropical geometry, projections which preserve smoothness, even at a single point are relatively rare and thus can not be considered to be the generic case. \square

3. CROSSING NUMBERS AND GONALITY

By Theorem 1.1, any trivalent tropical curve has a planar immersion. We can define the **tropical crossing number** of the curve to be the minimal number of nodes, counted with multiplicity, of any planar immersion. Thus, the tropical curve has a smooth planar embedding if and only if its crossing number is zero, and Theorem 1.2 establishes a quadratic upper bound on the crossing number. In this section, we establish a lower bound on the crossing number of an abstract tropical curve in terms of its genus and divisorial gonality. We use this to prove Theorem 1.3 in the form of Corollary 3.7.

A tool we will repeatedly use in this section and the next is the dual subdivision of a plane curve. We gather some basic facts about the dual subdivision which will come up in many of the proofs. Since any nodal tropical curve Γ is balanced, [RGST05, Thm. 3.3] states that it is the non-differentiable locus of a concave piecewise-linear function, whose slopes are

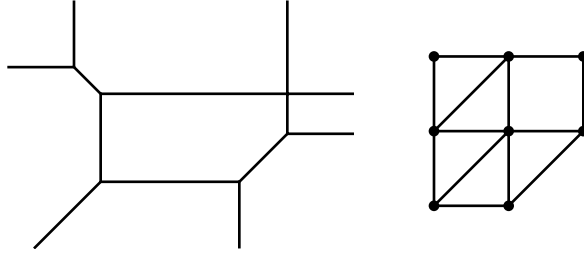


FIGURE 3. A plane tropical curve on the left and the corresponding dual subdivision on the right. The triangles of the subdivision correspond to the trivalent vertices of the curve and the square represents the curve's unique node.

integral and such that the difference between the slope vectors on either side of an edge of Γ has relatively prime entries. The dual subdivision Δ is the projection of the lower convex hull of the slopes of the piecewise linear function. The slopes on either side of an edge of Γ will be joined by an edge of this subdivision and since the difference vector has relatively prime entries, the only integral points contained in the edge are its endpoints. See [MS15, Sec. 1.3] for details on the dual subdivision.

For us, the relevance of the above construction will be the duality between the curve and the subdivision, in which the vertices and edges of the curve correspond to the polygons and edges of the subdivision respectively. Likewise, the bounded and unbounded regions of the complement of the curve correspond to the vertices in the interior and on the boundary of Δ . An example of a tropical curve and its dual subdivision are shown in Figure 3.

Definition 3.1. The **genus** g of an abstract tropical curve is the rank of the first homology of its underlying graph, i.e. $\dim_{\mathbb{Q}} H_1(\Gamma, \mathbb{Q})$. \square

Proposition 3.2. *If Γ is a tropical curve of genus g and $\iota(\Gamma)$ is an immersion with n nodes, counted with multiplicities, then*

$$i = g + n,$$

where i is the number of integral points in the interior of the dual polygon.

We will use the following well-known result about the number of lattice points in an integral polygon.

Proposition 3.3 (Pick's Theorem). *Let P be a polygon with integral vertices. If P has area A and i lattice points, of which b are on the boundary of the polygon, then we have the relation $A = i - b/2 - 1$.*

Proof of Prop. 3.2. We first introduce some additional notation. We write g' for the genus of $\iota(\Gamma)$, without resolving any nodes, and n' for the number of such nodes, without any multiplicities. Since the procedure for resolving a node adds a vertex, but does not change the number of edges, we have that

$g = g' - n'$. Therefore, it will suffice to prove the relation

$$(2) \quad i = g' + n - n'.$$

The dual subdivision Δ will have a triangle corresponding to each trivalent vertex and a parallelogram corresponding to each node. We now count how the interior lattice points of Δ fall within the subdivision. Because of our local smooth model, the only lattice points in a triangle will be its vertices. Moreover, as noted above, there are no lattice points on the interior of any edge of the subdivision. Thus, an internal lattice point of Δ is either a vertex of the subdivision or in the interior of a parallelogram. Each vertex of the former category defines a bounded region of the complement of $\iota(\Gamma)$ and thus there are g' such vertices. Therefore, we will have verified (2) if we can show that a node of multiplicity m is dual to a parallelogram containing $m - 1$ interior lattice points.

Let P be such a parallelogram. A simple computation shows that the area of P is equal to the multiplicity m of the node as in (1). As noted above, the only lattice points on the boundary of P are the 4 vertices. Therefore, Pick's formula (Prop. 3.3) tells us that P contains $m + 3$ lattice points and thus $m - 1$ interior lattice points. \square

Our method for proving the asymptotic lower bound on the crossing number from Theorem 1.3 is the divisorial gonality of graph as in [Bak08]. Specifically a graph Γ is defined to have **divisorial gonality** d if d is the least integer such that there exists a divisor on Γ of degree d and rank 1. We refer to [GK08, Sec. 1] or [MZ08, Def. 7.1] for the definitions of the rank of a divisor and to [MS15, Sec. 3.6] for the definition of stable intersection. If D is a divisor on Γ , meaning a formal sum of points, then we define $\iota_*(D)$ to be the formal sum of points in \mathbb{R}^2 by taking the images of the points of D . This construction relates the divisor theory of the abstract tropical curve with stable intersections of the immersion by the following lemma.

Lemma 3.4. *Let a be a real number and let L be the line in \mathbb{R}^2 defined by $x = a$. If $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ is the piecewise linear function $\phi(x) = \max\{x - a, 0\}$, then $\phi \circ \iota$ is a piecewise linear function with integer slopes, so that it has a divisor $\text{div}(\phi \circ \iota)$, such that $\iota_*(\text{div}(\phi \circ \iota))$ equals the stable intersection of L with $\iota(\Gamma)$.*

Proof. By [MS15, Def. 3.6.5], the stable intersection consists of those points contained both in L and in a segment of $\iota(\Gamma)$ which is not parallel to L . Let p be such a point and we compute the multiplicity of the stable intersection at p using a small perturbation of L to the right so that the multiplicity is a summation over segments of $\iota(\Gamma)$ containing both p and also extending to the right of L . Fix one such segment and let $v = (v_1, v_2)$ be the minimal integral vector parallel to it. Then, the contribution of this segment to the multiplicity is the index of the lattice generated by $(0, 1)$ and v in \mathbb{Z}^2 by [MS15, Def. 3.6.5]. The former lattice can also be generated by $(0, 1)$ and $(e_1, 0)$, so the index is e_1 .

On the other hand, $\phi \circ \iota$ is non-zero to the right of L and so its divisor can also be computed by taking the slopes along the segments which extend to the right of L . If e is again the minimal integral vector parallel to such a segment, then, by our definition of edge lengths, the slope is $\phi(p + e) - \phi(p) = e_1$, which is in integer. Thus, the two multiplicity computations coincide, and so we have the desired equality from the lemma statement. \square

Proposition 3.5. *Let $\iota: \Gamma \rightarrow \mathbb{R}^2$ be an immersion of a tropical curve in the plane and suppose that $\iota(\Gamma)$ is not just a line. Then the stable intersection of $\iota(\Gamma)$ with a straight line of rational slope defines a divisor with rank at least 1.*

Proof. We first change coordinates so that the line L is parallel to the y -axis, say defined by $x = a$. Let D be the divisor formed by the stable intersection of Γ with L . Then we need to show that for any point $p \in \Gamma$, there exists an effective divisor linearly equivalent to D which contains p . We first let L' be the vertical line containing $\iota(p)$, and say that L' is defined by $x = a'$. Let D' the stable intersection of L' with $\iota(\Gamma)$. We define $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ to be the piecewise linear function

$$\max\{x - a', 0\} - \max\{x - a, 0\}.$$

Then, using Lemma 3.4 and the linearity the div function, $D' = D + \text{div}(\phi \circ \iota)$, so D' is linearly equivalent to D . Moreover, if p is an isolated point of $L' \cap \iota(\Gamma)$, then p is in D' as required.

Otherwise, p is contained in a positive-length interval of $L' \cap \iota(\Gamma)$. If this interval is bounded, then, by the local model of a smooth curve, both its endpoints will be contained in the stable intersection D' . These endpoints can be moved together the same distance so that one of them contains p . If the interval is unbounded, then by our assumption, it must still be bounded in one direction, and D' will contain that endpoint. We can then move the point of D' along the unbounded edge until it contains p . \square

Proposition 3.6. *If Γ is an abstract tropical curve with divisorial gonality $d > 2$ and genus g , then the tropical crossing number of Γ is at least $\frac{3}{8}(d - 2)^2 - g + \frac{1}{2}$.*

Our proof uses the same technique as in [Smi15, Thm. 3.3], which bounds the gonality of a curve in a smooth toric variety.

Proof. Suppose that Γ has divisorial gonality d and we have a planar immersion. By [RGST05, Thm. 3.3], Γ is dual to a subdivision of a Newton polygon Δ . Recall that the **lattice width** w of the polygon Δ is the smallest possible value of $\max\{\lambda \cdot \mathbf{x} \mid \mathbf{x} \in \Delta\} - \min\{\lambda \cdot \mathbf{x} \mid \mathbf{x} \in \Delta\}$ as $\lambda \in \mathbb{Z}^2$ ranges over all non-zero integer vectors. Then, applying Proposition 3.5 to the line defined by $\lambda \cdot \mathbf{x} = 0$, we see that the divisorial gonality of Γ is at most w , i.e. $w \geq d$. We let $\Delta^{(1)}$ denote the convex hull of the interior points of Δ and let w' denote the lattice width of $\Delta^{(1)}$. Then Theorem 4 from [CC12] shows that $w' = w - 2$ or Δ and $\Delta^{(1)}$ are unimodular simplices scaled by w and

$w - 3$ respectively. We first deal with the former case, for which we have $w' \geq d - 2$.

Then, Theorem 2 from [FTM74] implies that A , the area of $\Delta^{(1)}$ is at least $3(d - 2)^2/8$. By our assumption that $d > 2$, we know that $w' > 0$, so $\Delta^{(1)}$ is not contained in a line. We thus apply Pick's formula, Proposition 3.3, to get the relation $i = A + b/2 + 1$, where i is the number of integral points in $\Delta^{(1)}$, and b is the number of those integral points on the boundary of $\Delta^{(1)}$. Thus,

$$i \geq A + 1 \geq \frac{3(d - 2)^2}{8} + 1.$$

We now return to the case when $\Delta^{(1)}$ is a unimodular simplex scaled by $w - 3$. We can directly compute that in this case, with i again the number of integral points in $\Delta^{(1)}$,

$$i = \frac{(w - 2)(w - 1)}{2} \geq \frac{(d - 2)(d - 1)}{2} = \frac{(d - 2)^2}{2} + \frac{d - 2}{2} \geq \frac{3(d - 2)^2}{8} + \frac{1}{2}.$$

Thus, in either case, we can apply Proposition 3.2 to get that the number of nodes is $i - g$ and thus at least $\frac{3}{8}(d - 2)^2 - g + \frac{1}{2}$, as claimed. \square

We prove Theorem 1.3 using the chain of loops with bridges from [JP14], which is similar to the chain of loops from [CDPR12], but with bridges added between the loops. In particular, these graphs are planar. As in [CDPR12] and [JP14], we will assume that the lengths of the edges in the loops are generic, for which it is sufficient for the vector of length assignments for edges in the loops to avoid a finite union of rational hyperplanes. The following then shows that this family is far from having planar embeddings for large g .

Corollary 3.7. *If Γ is the chain of $g \geq 3$ loops with bridges and has generic edge lengths then the crossing number of Γ is at least $3g^2/32 - 11g/8 + 7/8$. In particular, the crossing number of this family of graphs is quadratic in the number of edges.*

Proof. By [CDPR12, Thm. 1.1] and the fact that divisorial gonality does not change by adding bridges [JP14, Rmk. 1.2], we deduce that Γ is Brill-Noether general. This implies that it has divisorial gonality $\lceil g/2 \rceil + 1$. In particular, when $g \geq 3$, the divisorial gonality is greater than 2. Substituting this into the expression in Proposition 3.6 and dropping the ceiling, we see that Γ has crossing number at least $3g^2/32 - 11g/8 + 7/8$.

The last sentence follows from the fact that Γ has $3g - 3$ edges and thus the crossing number is also quadratic in the number of edges. \square

4. LOW GENUS CURVES

In this section, we examine the crossing numbers of tropical curves with genus at most 2. We begin with genus 0. The underlying graph of a genus 0 tropical curve is a tree and in Proposition 4.3, we characterize genus 0 tropical curves with crossing number 0 in terms of its underlying graph. In genus 1, we only consider graphs consisting of infinite edges attached to the central

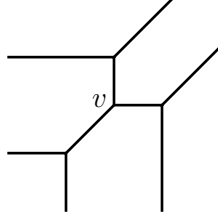


FIGURE 4. A smooth tropical curve whose underlying graph structure is the windmill graph.

loop, in which case Proposition 4.4 gives a sharp bound on the crossing number. For genus 2 curves, we restrict to **stable tropical curves**, by which we mean that every vertex has degree equal to 3. In particular, stable tropical curves have no infinite edges. Proposition 4.7 gives the crossing numbers of stable genus 2 tropical curves. In this case, the tropical crossing number depends on the edge length.

The following proposition is also proved as Proposition 8.3 in [BJMS15], where the curves satisfying the hypothesis are called sprawling.

Proposition 4.1. *Let Γ be a trivalent abstract tropical curve and let v be a degree 3 vertex of Γ such that removing v disconnects Γ into three components A , B , and C . We consider each of these components to be a tropical curve by including v in each of them.*

Suppose that A , B , and C each contain at least one trivalent vertex. Then Γ has a planar embedding if and only if A , B , and C each contain a single trivalent vertex, in which case Γ looks like the curve shown in Figure 4.

Proof. Consider a planar embedding of Γ . By making a change of coordinates, we may assume that the outgoing edges from v have directions $(1, 0)$, $(0, 1)$, and $(-1, -1)$. Let w_A , w_B , and w_C be the first trivalent vertices in these respective directions, which we've assumed to exist. We will show that v , w_A , w_B , and w_C are the only trivalent vertices.

The local model for a smooth curve implies that the outgoing directions at w_A are $(-1, 0)$, $(-a, -1)$, and $(a + 1, 1)$ for some $a \in \mathbb{Z}$. Similarly, the outgoing directions at w_B are $(0, -1)$, $(-1, -b)$, and $(1, b + 1)$ for some $b \in \mathbb{Z}$ and at w_C they are $(1, 1)$, $(c, c - 1)$, and $(-c - 1, -c)$ for some $c \in \mathbb{Z}$.

Consider the component of the complement $\mathbb{R}^2 \setminus \Gamma$ which lies between w_A and w_B . This component will be convex because the angle between any two rays of the standard smooth model is less than 180° , even after any change of coordinates. Furthermore, because the only paths between w_A and w_B pass through v , this component must be unbounded. On the other hand, we have the edge from w_A in the direction $(a + 1, 1)$, so in particular, with increasing y -coordinate, and we have an edge from w_B with slope $b + 1$, so if $b + 1 \leq 0$, then these two directions would eventually meet, and thus they could not form edges of a convex, unbounded region. We conclude that $b \geq 0$.

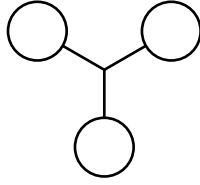


FIGURE 5. The lollipop abstract tropical curve, which does not have a planar embedding by Corollary 4.2.

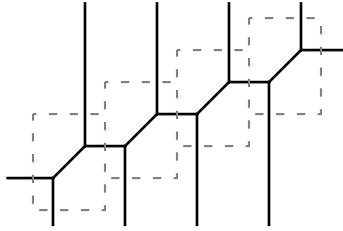


FIGURE 6. Embedding of the caterpillar graph with 10 leaves. Caterpillar graphs with more leaves can also be embedded by adding more repetitions of the blocks in the dashed lines.

Applying the same analysis to the region between w_C and w_B , we get that $b \leq 0$, so $b = 0$ and by symmetry $a = c = 0$ as well.

At this point, we know that, in a neighborhood of v , w_A , w_B , and w_C , the embedding of Γ must look like in Figure 4. However, the region between w_A and w_B now has two edges parallel to $(1, 1)$ in its boundary, and as we've seen, this region must be unbounded, so these edges are also unbounded. By symmetry, each of the vertices w_A , w_B , and w_C must have two unbounded edges from it. Thus, there are no further trivalent vertices, which is what we wanted to show. \square

Corollary 4.2. *The “lollipop curve” shown in Figure 5 has crossing number at least 1, for any lengths.*

Corollary 4.2 is a strengthening of the last sentence of Proposition 2.3 from [BLMPR14]. In that paper, they studied smooth plane quartics and found examples of all combinatorial types of genus 3 graphs except for the lollipop graph. Corollary 4.2 further shows that the lollipop graph does not have a planar embedding, even if the curve is not required to be quartic.

Proposition 4.3. *Let Γ be a genus 0 abstract tropical curve. Then Γ has crossing number 0 if and only if the underlying graph is a subdivision of the caterpillar graph or the windmill graph.*

Proof. In the case of the caterpillar graph or the windmill graph, the embedding can be constructed as in Figures 6 and 4 respectively. Conversely, if the tree Γ does have a planar embedding, then we can apply Proposition 4.1 at

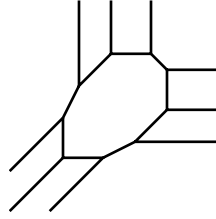


FIGURE 7. Embedding of the sun curve with 9 or fewer infinite edges.

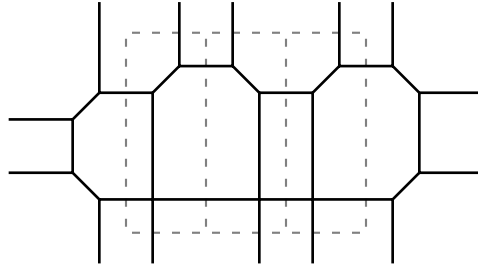


FIGURE 8. Immersion of the sun curve with more than 9 infinite edges. In the example depicted, the immersed tropical curve has 14 infinite edges and thus 3 crossings. The number of crossings can be varied by adding or removing blocks of the form shown in the dotted boxes. Each such block adds two infinite edges and one crossing.

each trivalent vertex, and there are two possibilities. If, for some vertex v , all components of $\Gamma \setminus \{v\}$ have only a single trivalent vertex, then Γ is a windmill graph. On the other hand, if for all trivalent vertices v , at least one of the components of $\Gamma \setminus \{v\}$ has no trivalent vertices, then Γ is a caterpillar graph. \square

Proposition 4.4. *Let Γ be a curve whose underlying graph is a sun: a cycle with n edges attached to the cycle. Then the crossing number of Γ is at least $\lceil n/2 \rceil - 4$ if $n > 9$ and this bound is sharp for some edge lengths.*

Proof. If we have an immersion of Γ with k nodes, then the dual triangulation will be a polygon with $k + 1$ interior lattice points by Proposition 3.2. Since each ray of the sun graph will produce an unbounded edge, the polygon has at least n edges and so at least n lattice points on its boundary. Scott's inequality is a bound on the number b of lattice points on the boundary of a polygon in terms of the number in the interior lattice points [Sco76] (see also [HS09]). When $n > 9$, so that $b \geq n > 9$, Scott's inequality has the form $b \leq 2(k + 1) + 6$. Since $n \leq b$, we can then solve for k to get the relation $k \geq n/2 - 4$, from which the desired inequality follows because the crossing number must be an integer.

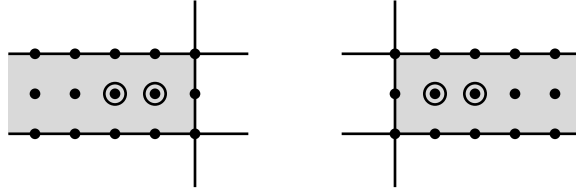


FIGURE 9. Lemma 4.6 says that the Newton polygon can be constrained to one of these two shaded regions. In either case, the interior points of the Newton polygon are the circled dots, which have coordinates $(0,0)$ and $(1,0)$.

To show that this bound can be sharp, we give examples of immersions of tropical curves which achieve them. For $n \leq 9$, the embedding in Figure 7 justifies the requirement of $n > 9$ from the proposition statement. On the other hand, the pattern in Figure 8 with k of the dotted blocks will give an immersion with k crossings and $2k + 8$ infinite edges. For $n > 9$, by taking $k = \lceil n/2 \rceil - 4$, we have an example which has n or $n + 1$ infinite edges and achieves the crossing number bound from the proposition statement. \square

Proposition 4.5. *Let Γ be the theta graph as in the right of Figure 1, with all edge lengths equal. Then Γ has crossing number 1.*

The first step in the proof of Proposition 4.5 is the following Lemma 4.6, which constrains the possible shapes of the Newton polygon of an embedding of the theta graph, or of any other genus 2 graph. The possibilities in the conclusion of Lemma 4.6 are illustrated in Figure 9.

Lemma 4.6. *Let Δ be the Newton polygon dual to a smooth embedding of an abstract tropical curve of genus two. Then we can choose an affine change of coordinates such that the interior points of Δ are $(0,0)$ and $(1,0)$ and such that either of the following inequalities hold:*

- (1) *The points of Δ are bounded by $-1 \leq y \leq 1$ and $x \leq 2$.*
- (2) *The points of Δ are bounded by $-1 \leq y \leq 1$ and $x \geq -1$.*

Moreover, the transformation changing from coordinates satisfying (1) to those satisfying (2) acts as the identity on the $y = 0$ line.

Proof. Since the embedding, by definition, has no nodes, Proposition 3.2 implies that Δ has exactly two integral points in its interior. After an affine change of coordinates which is common to both (1) and (2) from the statement, we can assume that these two interior points are $(0,0)$ and $(1,0)$. Now suppose that there is a vertex of the polygon (a,b) with $b > 1$. We claim that the integral point $v = (\lceil a/b \rceil, 1)$ would also be in the interior of Δ , which would be a contradiction. To see this claim, let $r = \lceil a/b \rceil - a/b$, and note that $0 \leq r \leq 1 - 1/b$. We can then write:

$$v = \left(1 - r - \frac{1}{b}\right) (0,0) + r(1,0) + \left(\frac{1}{b}\right) (a,b),$$

which is a convex linear combination by the previously noted inequalities. Since $(0, 0)$, $(1, 0)$, and (a, b) are all in Δ , so is v . Moreover, the former two points are in the interior of Δ , and v is not equal to (a, b) , so v is in the interior of Δ . Thus, we've proved the desired claim, and so there can be no interior lattice point (a, b) with $b > 1$. By symmetry, there are no vertices (a, b) in Δ with $b < -1$ either, and so we've proved the first half of either set of bounds.

We now turn to finding bounds on the horizontal extent of Δ . Suppose that $(a, 1)$ is the rightmost vertex of Δ on the $y = 1$ line. We now change coordinates by sending (x, y) to $(x - (a - 2)y, y)$, after which $(2, 1)$ is the rightmost such point. We now claim that all points in Δ are bounded by $x \leq 2$. On the $y = 1$ line, this is by our change of coordinates, and if there exists a vertex (a, b) with $a > 2$ and either $b = 0$ or $b = -1$, then the vertex $(2, 0)$ would be in the interior of Δ , which is a contradiction. Therefore, we've found coordinates satisfying the bounds in (1).

To find the second set of coordinates, we start with the leftmost vertex on the $y = 1$ and apply an analogous change of coordinates, which is the identity on the $y = 0$ line and after which Δ is bounded by $x \geq -1$, as desired. \square

Proof of Proposition 4.5. The immersion from Example 2.7 shows that the crossing number is at most 1, and so it remains to show that there is no planar embedding of Γ .

We assume for the sake of contradiction that we have a planar embedding ι . We consider the dual triangulation of the Newton polygon Δ , for which we first assume we have coordinates for which Δ is bounded as in Lemma 4.6(1). Since Γ is a theta graph, there must be an edge of the embedding separating the two bounded regions. Dually, the bounded regions correspond to the points $(0, 0)$ and $(1, 0)$, so there must be an edge of the triangulation between them, which then corresponds to a vertical edge in $\iota(\Gamma)$. The triangles above and below the edge joining $(0, 0)$ and $(1, 0)$ correspond to the two trivalent vertices of Γ . We label the edges of Γ as e_1 , e_2 , and e_3 , such that $\iota(e_2)$ is the vertical edge, and the regions to the left and right of this edge are bounded by $\iota(e_1 \cup e_2)$ and $\iota(e_2 \cup e_3)$ respectively.

Now we consider the subset e'_3 of e_3 consisting of points p of e_3 such that $\iota(p) + (\epsilon, 0)$ is in an unbounded region or unbounded ray for all sufficiently small ϵ . Equivalently, $\iota(e'_3)$ is the union of the segments of $\iota(e_3)$ whose dual in the triangulation are edges connecting $(1, 0)$ and a point with first coordinate greater than 1. By the inequalities of Lemma 4.6(1), the only possibilities for the second point are $(2, 1)$, $(2, 0)$, and $(2, -1)$, which correspond to edges in $\iota(e'_3)$ parallel to the vectors $(1, -1)$, $(0, -1)$ and $(-1, -1)$, respectively. All of these vectors have -1 in the second coordinate, so the total length of the segment e'_3 equals the height of $\iota(e'_3)$. Since the length of e_3 equals the length of e_2 , which equals the height of the vertical segment $\iota(e_2)$, we know that e'_3 must consist of all of e_3 . As a consequence, the only possible endpoints of edges of the triangulation containing $(1, 0)$ are $(0, 0)$, $(2, 1)$, $(2, 0)$, and

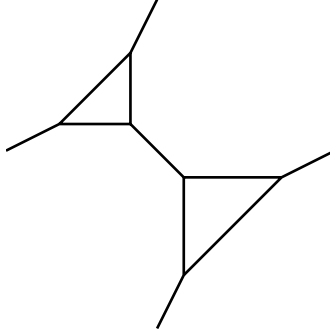
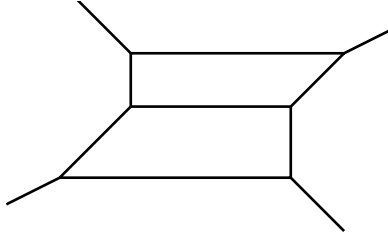


FIGURE 10. Embedding of the barbell graph.


 FIGURE 11. Embedding of the theta graph when $a < b \leq c$.
In this figure, the middle edge has length a .

$(2, -1)$. Thus, the triangles above and below the edge from $(0, 0)$ to $(1, 0)$ in the triangulation must contain the vertices $(2, 1)$ and $(2, -1)$ respectively. The midpoint of these two vertices will be $(2, 0)$ and note that this midpoint is preserved under linear changes of coordinates which also preserve the $y = 0$ line.

Second, we consider coordinates such that Δ is bounded as in Lemma 4.6(2). By symmetry, the same argument applied to e_1 shows that, in these coordinates, the triangles above and below the edge between $(0, 0)$ and $(1, 0)$ have their third vertices at $(-1, 1)$ and $(-1, -1)$, respectively. The midpoint of these two vertices is $(-1, 0)$, which would remain true when changing to the coordinates as in Lemma 4.6(1). Therefore, we have a contradiction with the previous paragraph, so there is no embedding of the graph Γ , so its crossing number is 1. \square

Proposition 4.7. *Let Γ be a stable tropical curve of genus two. Then Γ has tropical crossing number 0 unless Γ is the theta graph in Figure 1 with all edge lengths equal, in which case it has crossing number 1.*

Proof. There are two combinatorial types of trivalent graphs of genus 2. In the case of the barbell graph, we can take the planar embedding shown in Figure 10 for all possible edge lengths. For the theta graph in Figure 1, there are three possibilities depending on the edge lengths a , b , and c . By

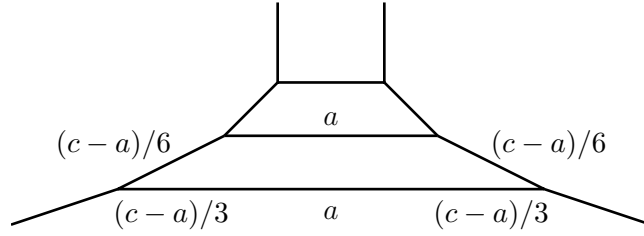


FIGURE 12. Embedding of the theta graph with edge lengths a , b , and c when $a = b < c$.

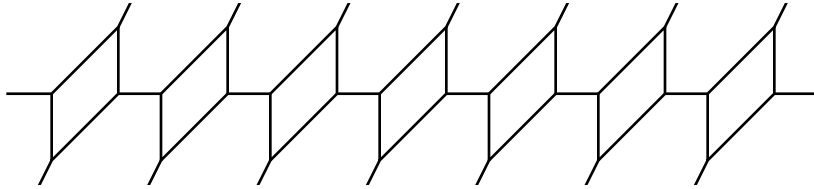


FIGURE 13. Embedding without crossings of a chain of 7 loops when both halves of each loop have the same length. The same pattern can be continued to give an embedding if the graph is extended to have arbitrarily many loops.

symmetry, we can assume that $a \leq b \leq c$. If we have a strict inequality $a < b$ or if $a = b < c$, then we can use the embeddings in Figures 11 and 12 respectively. Otherwise, all of the edge lengths are equal, and the crossing number is 1 by Proposition 4.5. \square

Remark 4.8. The example of the theta graph in Proposition 4.7 shows that the crossing number can increase for specializations of the metric parameters, i.e. the crossing number function is not lower semi-continuous. Moreover, it is not upper semi-continuous because the crossing number can jump down in specializations as well. For example, the chain of $g \geq 15$ loops with generic edge lengths has positive crossing number by Corollary 3.7, but if all edge lengths are equal then it is planar by the embedding shown in Figure 13. Also, a consequence of the proof of Proposition 4.4 is that any embedding of a sun curve with 9 infinite edges is equivalent to that in Figure 7, up to change of coordinates in $GL_2(\mathbb{Z})$. However, such an embedding implies non-trivial conditions among the lengths of the edges of the cycle. \square

For curves of higher genus, we refer to [BJMS15, Sec. 5–8], where curves of crossing number 0 are characterized using computational techniques. In particular, they show that it is feasible to enumerate the possible Newton polygons of low genus curves and from these compute the defining inequalities of the tropical curves which can then arise.

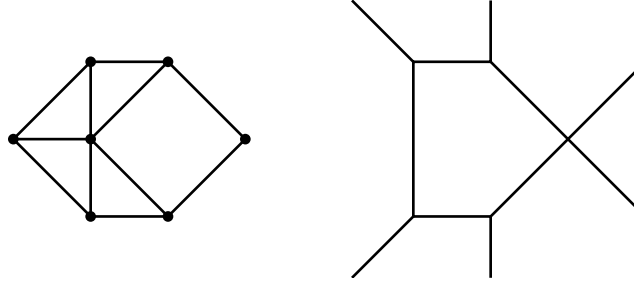


FIGURE 14. On the right is a nodal tropical curve which is the totally faithful tropicalization of the nodal rational curve given in Example 5.2. On the left is the dual subdivision of the Newton polygon for this curve.

5. ALGEBRAIC REALIZATIONS OF TROPICAL CURVES

In this section, we consider applications of our results to realizations of tropical curves by algebraic curves. We recall that for any curve $C \subset \mathbb{G}_m^2$ over a field K with valuation, the tropicalization of C is a union of finitely many edges in \mathbb{R}^2 . One characterization of the tropicalization $C \subset \mathbb{G}_m^2$ is as the projection of the Berkovich analytification C^{an} using the valuations of the coordinate functions of \mathbb{G}_m^2 . We are interested in cases when this map preserves the skeleton of the analytification in the following sense:

Definition 5.1. Let $C \subset \mathbb{G}_m^2$ be a nodal curve over a field K with non-trivial valuation and let \tilde{C} be the normalization of C . We say that C has **totally faithful tropicalization** if there exists a finite set of points $p_1, \dots, p_n \in \mathbb{R}^2$ such that:

- (1) The skeleton of \tilde{C} maps isometrically onto the tropicalization of C , except at p_1, \dots, p_n .
- (2) At each point p_i , the tropicalization of C is a node, as in Definition 2.5, and the number of nodes of C tropicalizing to p_i is equal to the multiplicity of the node at p_i , as in (1). \square

Definition 5.1 is a generalization to nodal curves of the definition given in [CFPU14]. Baker, Payne, and Rabinoff have shown that any algebraic curve C has a faithful tropicalization in the sense that there's an open subset $C' \subset C$ and an embedding $C' \rightarrow \mathbb{G}_m^n$ such that the map from the analytification $(C')^{\text{an}}$ is an isometry on the skeleton of C , but this is not necessarily a totally faithful tropicalization because the skeleton of C' is larger than that of C when $C' \subsetneq C$. Theorem 1.4 constructs totally faithful tropicalizations, but working in reverse, beginning with an abstract tropical curve and constructing the algebraic curve.

Example 5.2. Let $K = \mathbb{C}[[t]]$ and let C be the curve in \mathbb{G}_m^2 defined by the equation:

$$(1+t)^2 t^2 x^3 y + t x^2 y^2 + (3t^4 + 3t^3 + 2t + t + 1)x^2 y - x y^2 \\ + t x^2 + (3t^3 + t - 2) t x y - x + (1-t)^2 (1+t+t^2) y.$$

The dual subdivision of the Newton polygon and the corresponding tropical curve are shown in Figure 14. The tropical curve is nodal with a single node of multiplicity 2 at the point $(0, 0)$. While a generic curve with this Newton polygon would have genus 2, one can check that C has nodal singularities at the two points:

$$((1-t)/(1+t), 1) \quad \text{and} \quad ((-1-t)/(1+t), -1),$$

and therefore, C is rational. Both of these singularities have coordinates with valuation 0 and therefore lie above the multiplicity 2 node in the tropicalization. Therefore, the nodal curve C has totally faithful tropicalization. \square

Proof of Theorem 1.4. By Theorem 1.1, Γ admits a planar immersion $\iota: \Gamma \rightarrow \mathbb{R}^2$ and by Remark 2.8, we can assume that the vertices in this immersion have rational coordinates. Then, it will be sufficient to show that this nodal plane curve is the tropicalization of some algebraic curve, for which we use Mikhalkin's correspondence theorem [Mik05], and more specifically the algebraic version due to Shustin [Shu05, Thm. 3]. While the cited theorem only asserts that a certain count of nodal algebraic curves equals a weighted count of the corresponding tropical curves, the proof works by constructing at least one algebraic curve for each nodal tropical curve. In particular, [Shu05, Sec. 3.7] shows that for any nodal tropical curve $\iota(\Gamma)$, it is possible to find certain auxiliary data, denoted by S , F , and R in that paper. Then, [Shu05, Lem. 3.12] states that from $\iota(\Gamma)$, together with this auxiliary data, we can find a nodal algebraic curve tropicalizing to $\iota(\Gamma)$. \square

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